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GEOMETRIC STIFFNESS CHARACTERISTICS OF A ROTATING ELASTIC APPENDAGE†

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INTRODUCTION

In [1] there appear equations of motion which characterize the small time-varying deformations of a distributed–mass finite element model of an elastic appendage attached to a rigid body having arbitrary motions. Reference [2] provides the equations of motion of a dynamical system of interconnected rigid bodies, each of which has attached to it a nonrigid appendage. In concert these two references establish the basis for a generic digital computer program to be developed for the simulation of nonrigid spacecraft. The purpose of this note is to strengthen [1] by one subtle but significant generalization and one correction and elaboration.

A GENERALIZATION

As shown in [1], (see equation (164), with damping excluded), if one assumes a distributed–mass, finite–element model with mass present also in the form of rigid bodies concentrated at each node, and chooses to characterize the unknowns as the $6n$ small linear and angular deformational displacements of the n rigid nodal bodies relative to some nominal state, and assembles these in the $6n$ by 1 column matrix q, then the ordinary differential equations of appendage vibratory deformation have the form

$$M'\ddot{q} + G'\dot{q} + K'q + A'q = L' \tag{1}$$

where M' and K' are symmetric and G' and A' are skew symmetric matrices. If the base to which the appendage is attached rotates at a constant rate about an inertially fixed axis, then the coefficient matrices in equation (1) are constant, and $L' = 0$.

It is important in some cases to recognize that the steady state stresses in a rotating elastic system can contribute to the skew-symmetric matrix A' by means of an *asymmetric* "geometric stiffness matrix," and that the result can be the elimination of the troublesome matrix A'. These possibilities are precluded in [1] by the seemingly insignificant assumption that nodal body incremental rotations are sequential rotations about *permanently orthogonal axes*. As a consequence of this assumption, the generalized force Q_α corresponding to a nodal body rotation $\beta_\alpha{}^j$ of the jth body is the $\underline{a}_\alpha$ component of the torque $\underline{T}^j$ applied to the jth nodal body, since by first principles

$$Q_\alpha = \underline{T}^j \cdot \frac{\partial \underline{\omega}^j}{\partial \dot{\beta}_\alpha} = \underline{T}^j \cdot \underline{a}_\alpha = T_\alpha{}^j \tag{2}$$

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In the general case, $\partial \underline{\omega}^j / \partial \dot{\beta}_\alpha \neq \underline{a}_\alpha$, and one must make a distinction between Q_α and $T_\alpha{}^j$. (An example of this kind is shown in the following section.) The matrix designated $\bar{L}$ in [1] can always be interpreted as the matrix of generalized forces; only for the special case treated explicitly in Ref. [1] is the interpretation of $\bar{L}$ as a matrix of scalar components of force and torque for orthogonal axes (as in equation (19) of [1]) a valid interpretation. Thus we can broaden the scope of [1] (to include for example the problem in the following section) simply by extending the meaning of $\bar{L}$, and establishing for each problem a specific relationship between Q_α and $T_\alpha{}^j$. A possible implication of this generalization for the geometric stiffness matrix is established in the example following.

ILLUSTRATION OF ASYMMETRIC GEOMETRIC STIFFNESS MATRIX

Consider the rigid body B supported in a rotating housing body A by means of spring-mounted massless gimbals B' and A', as shown in Fig. 1. Note the dextral orthogonal sets of unit vectors of corresponding labels in the figure (e.g. $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ and $\underline{b}_1'$, $\underline{b}_2'$, $\underline{b}_3'$). Imagine that there exists a steady–state motion for which B maintains a fixed relationship to A, while the mass center C of B remains fixed in inertial space and A maintains the constant inertial angular velocity $\underline{\Omega}$, fixed somewhere in A but not parallel to $\underline{a}_1$, $\underline{a}_2$, or $\underline{a}_3$. Imagine further that in this steady state all unit vectors of like index are aligned, so that the gimbal hinge axes are orthogonal. In this state B is rotating at a constant rate about a nonprincipal axis, so that a body-fixed torque must be applied to B by means of the elastic springs at the three gimbal hinge axes parallel to $\underline{a}_1 \equiv \underline{a}_1'$, $\underline{a}_2' \equiv \underline{b}_2'$, and $\underline{b}_3' \equiv \underline{b}_3$. Rotations of the

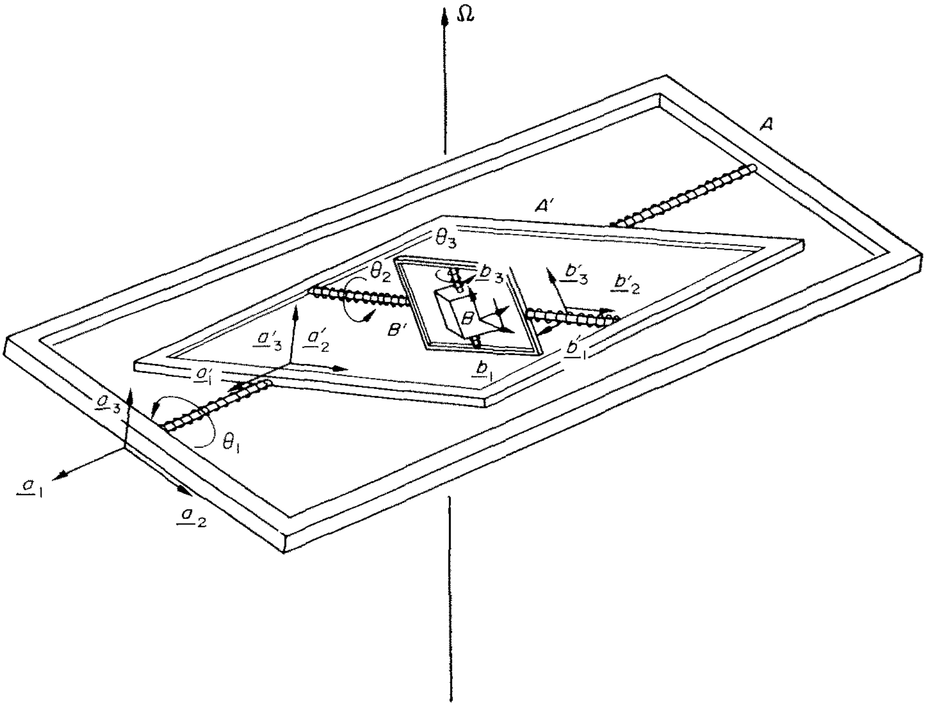

Fig. 1. Rotating body B with elastic rotation constraints.

corresponding angles from the unstressed state to the proposed steady state are designated Δ_1, Δ_2, and Δ_3, and the corresponding spring constants are k_1, k_2, and k_3, so that in the steady state the torque applied to B is given by

$$\begin{aligned} \underline{T}_0 &= -k_1\Delta_1\underline{b}_1 - k_2\Delta_2\underline{b}_2 - k_3\Delta_3\underline{b}_3 \\ &= -k_1\Delta_1\underline{a}_1 - k_2\Delta_2\underline{a}_2 - k_3\Delta_3\underline{a}_3 . \end{aligned} \tag{3}$$

When body B is perturbed from its steady state orientation relative to A, the expression for the torque $\underline{T}$ applied to B becomes perhaps surprisingly complicated. If θ_1, θ_2, θ_3 are gimbal rotations from the steady state corresponding to axes parallel to $\underline{a}_1 \equiv \underline{a}_1'$, $\underline{a}_2' \equiv \underline{b}_2'$, and $\underline{b}_3' \equiv \underline{b}_3$ respectively, then the inertial angular velocity of B becomes

$$\underline{\omega} = \underline{\Omega} + \dot{\theta}_1\underline{a}_1 + \dot{\theta}_2\underline{a}_2' + \dot{\theta}_3\underline{b}_3' \tag{4}$$

and our immediate knowledge of $\underline{T}$ is limited to the observations

$$\begin{aligned} \underline{T}\cdot\underline{a}_1 &= -k_1(\Delta_1 + \theta_1) \\ \underline{T}\cdot\underline{a}_2' &= -k_2(\Delta_2 + \theta_2) \\ \underline{T}\cdot\underline{b}_3' &= -k_3(\Delta_3 + \theta_3). \end{aligned} \tag{5}$$

Although one can manipulate these expressions algebraically to obtain $\underline{T}$ in any vector basis, such as $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, present purposes are best served by calculating first the generalized forces

$$\begin{aligned} Q_1 &\triangleq \underline{T}\cdot\frac{\partial\underline{\omega}}{\partial\dot{\theta}_1} = \underline{T}\cdot\underline{a}_1 = -k_1(\Delta_1 + \theta_1) \\ Q_2 &\triangleq \underline{T}\cdot\frac{\partial\underline{\omega}}{\partial\dot{\theta}_2} = \underline{T}\cdot\underline{a}_2' = -k_2(\Delta_2 + \theta_2) \\ Q_3 &\triangleq \underline{T}\cdot\frac{\partial\underline{\omega}}{\partial\dot{\theta}_3} = \underline{T}\cdot\underline{b}_3' = -k_3(\Delta_3 + \theta_3). \end{aligned} \tag{6}$$

To obtain the matrix T representing $\underline{T}$ in vector basis $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$, we can define the matrices

$$T \triangleq \underline{T}\cdot\begin{bmatrix}\underline{a}_1\\ \underline{a}_2\\ \underline{a}_3\end{bmatrix}; \qquad \omega \triangleq \underline{\omega}\cdot\begin{bmatrix}\underline{a}_1\\ \underline{a}_2\\ \underline{a}_3\end{bmatrix};$$

$$Q \triangleq \begin{bmatrix}Q_1\\ Q_2\\ Q_3\end{bmatrix}; \qquad \theta \triangleq \begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}$$

and write

$$Q = \left(\frac{\partial\omega^T}{\partial\dot{\theta}}\right)T \tag{7}$$

finally inverting to obtain

$$T = \left(\frac{\partial \omega^T}{\partial \dot\theta}\right)^{-1} Q \equiv \begin{bmatrix} \frac{\partial \omega_1}{\partial \dot\theta_1} & \frac{\partial \omega_2}{\partial \dot\theta_1} & \frac{\partial \omega_3}{\partial \dot\theta_1} \\ \frac{\partial \omega_1}{\partial \dot\theta_2} & \frac{\partial \omega_2}{\partial \dot\theta_2} & \frac{\partial \omega_3}{\partial \dot\theta_2} \\ \frac{\partial \omega_1}{\partial \dot\theta_3} & \frac{\partial \omega_2}{\partial \dot\theta_3} & \frac{\partial \omega_3}{\partial \dot\theta_3} \end{bmatrix}^{-1} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix}. \tag{8}$$

In this case a little algebra provides

$$\begin{aligned} \underline{\omega} = \underline{\Omega} &+ \underline{a}_1(\dot\theta_1 + \dot\theta_3 \sin\theta_2) \\ &+ \underline{a}_2(\dot\theta_2 \cos\theta_1 - \dot\theta_3 \sin\theta_1 \cos\theta_2) \\ &+ \underline{a}_3(\dot\theta_3 \cos\theta_1 \cos\theta_2 + \dot\theta_2 \sin\theta_1) \end{aligned} \tag{9}$$

so that in the linear approximation

$$T \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix}^{-1} Q \tag{10}$$

or

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\theta_1 \\ -\theta_2 & \theta_1 & 1 \end{bmatrix} \begin{bmatrix} -k_1(\Delta_1 + \theta_1) \\ -k_2(\Delta_2 + \theta_2) \\ -k_3(\Delta_3 + \theta_3) \end{bmatrix} \cong \begin{bmatrix} -k_1(\Delta_1 + \theta_1) \\ -k_2(\Delta_2 + \theta_2) + k_3 \Delta_3 \theta_1 \\ k_1 \Delta_1 \theta_2 - k_2 \Delta_2 \theta_1 - k_3(\Delta_3 + \theta_3) \end{bmatrix}. \tag{11}$$

It is perhaps more illuminating to record this result in the form

$$T = -k_0 \Delta - k_0 \theta - k_\Delta \theta \tag{12}$$

where $\Delta \triangleq [\Delta_1 \quad \Delta_2 \quad \Delta_3]^T$ and

$$k_0 \triangleq \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}; \qquad k_\Delta \triangleq \begin{bmatrix} 0 & 0 & 0 \\ -k_3 \Delta_3 & 0 & 0 \\ +k_2 \Delta_2 & -k_1 \Delta_1 & 0 \end{bmatrix} \tag{13}$$

thereby revealing the asymmetric character of the "geometric stiffness matrix" k_Δ induced by the load existing in the springs in the steady state.

According to equation (60) of [1], the equation of rotation motion of B must be

$$T = \tilde\Omega I \Omega + I\ddot\theta + [\tilde\Omega I - (I\Omega)^\sim + I\tilde\Omega]\dot\theta + \{\tilde\Omega I \tilde\Omega - \tfrac{1}{2}[\tilde\Omega (I\Omega)^\sim + (I\Omega)^\sim \tilde\Omega] - \tfrac{1}{2}(\tilde\Omega I \Omega)^\sim\}\theta \tag{14}$$

where I is the inertia matrix of B in its own vector basis, $\Omega \triangleq [\underline{\Omega} \cdot \underline{a}_1 \; \underline{\Omega} \cdot \underline{a}_2 \; \underline{\Omega} \cdot \underline{a}_3]^T$, and the tilde operator has a significance illustrated by

$$\tilde\Omega \triangleq \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix}. \tag{15}$$

Thus it follows from the existence of $\theta \equiv 0$ as a steady state solution for equation (14) that

$$-k_0 \Delta = \tilde{\Omega} I \Omega \tag{16}$$

By scalar expansion of the expressions in equations (8) and (13), noting equation (10), we find the linear approximation

$$k_\Delta \theta \cong \left[\left(\frac{\partial \omega^T}{\partial \dot{\theta}}\right)^{-1} - U\right] k_0 \Delta$$

which with equation (16) becomes

$$k_\Delta \theta = \left[\left(\frac{\partial \omega^T}{\partial \dot{\theta}}\right)^{-1} - U\right] \tilde{\Omega} I \Omega. \tag{17}$$

It is with this interpretation that one must consider the final equations of vibration in the form

$$I\ddot{\theta} + [\tilde{\Omega} I - (I\Omega)^{\sim} + I\tilde{\Omega}]\dot{\theta} + \{\tilde{\Omega} I \tilde{\Omega} - \tfrac{1}{2}[\tilde{\Omega}(I\Omega)^{\sim} + (I\Omega)^{\sim}\tilde{\Omega}] + k_0 - \tfrac{1}{2}(\tilde{\Omega} I \Omega)^{\sim} + k_\Delta\}\theta = 0 \tag{18}$$

recognizing that the asymmetric form of k_Δ retrieves the possibility that the matrix coefficient of θ may be symmetric. For this illustrative example, one can extract from equation (16) the expression

$$\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \end{bmatrix} \triangleq \Delta = -k_0^{-1} \Omega I \Omega = \begin{bmatrix} (I_2 - I_3)\Omega_2 \Omega_3 / k_1 \\ (I_3 - I_1)\Omega_3 \Omega_1 / k_2 \\ (I_1 - I_2)\Omega_1 \Omega_2 / k_3 \end{bmatrix} \tag{19}$$

and combine this result with equation (13) to find the geometric stiffness matrix

$$k_\Delta = \begin{bmatrix} 0 & 0 & 0 \\ (I_2 - I_1)\Omega_1 \Omega_2 & 0 & 0 \\ -(I_1 - I_3)\Omega_3 \Omega_1 & (I_3 - I_2)\Omega_2 \Omega_3 & 0 \end{bmatrix}. \tag{20}$$

By expanding other terms in the coefficient matrix of θ in equation (18), one finds dramatic simplification, and equation (18) reduces to the form

$$I\ddot{\theta} + [\tilde{\Omega} I - (I\Omega)^{\sim} + I\tilde{\Omega}]\dot{\theta} + [\tilde{\Omega} I \tilde{\Omega} - \tilde{\Omega}(I\Omega)^{\sim} - (I\Omega)^{\sim}\tilde{\Omega} + k_0]\theta = 0. \tag{21}$$

Equation (21) has the classical form adopted by vibrating rotating systems, with the coefficients of θ and $\ddot{\theta}$ symmetric and the coefficient of $\dot{\theta}$ skew symmetric.

The importance of this example stems from its demonstration of the possibility of retrieving the symmetric form of the overall "stiffness matrix" in the final equation of vibration. This result is reassuring, since it conforms with the fact well-known in Lagrangian mechanics that it must be possible to structure the equations of motion of any linearized, conservative, holonomic system so as to obtain a symmetric coefficient–matrix for the generalized coordinates.

A CORRECTION FOR NONLINEARITIES

Reference [3] indicates the importance of retaining certain nonlinear terms in the strain–displacement equations for the determination of the stiffness characteristics of an elastic continuum vibrating relative to a deformed state. The second purpose of this addendum to [1] is to indicate that these nonlinear terms were incorrectly omitted in that development, and to show how these nonlinearities can in some cases contribute to the geometric stiffness matrix of the finite element model.

In [1] the 6 by 1 matrix of the element stresses induced by steady state rotation is denoted $\bar{\sigma}'$, and the corresponding strain matrix is called $\bar{\varepsilon}'$. The incremental (variational) stress and strain matrices are designated $\bar{\sigma}$ and $\bar{\varepsilon}$ respectively. Under the restriction to small strain (but without further restriction on deformational displacements), we can record the element strain energy $\mathscr{U}$ as

$$\mathscr{U} = \frac{1}{2}\iiint (\bar{\sigma}^T + \bar{\sigma}'^T)(\bar{\varepsilon} + \bar{\varepsilon}')\, dx\, dy\, dz$$

$$= \frac{1}{2}\iiint (\bar{\varepsilon}^T + \bar{\varepsilon}'^T)(\bar{\sigma} + \bar{\sigma}')\, dx\, dy\, dz \tag{22}$$

and the variational strain energy $\mathscr{U}^*$ (equation (21) of Ref. 1) would be

$$\mathscr{U}^* = \frac{1}{2}\iiint [\delta\bar{\varepsilon}^T(\bar{\sigma} + \bar{\sigma}') + (\bar{\varepsilon}^T + \bar{\varepsilon}'^T)\delta\bar{\sigma}]\, dx\, dy\, dz. \tag{23}$$

If now we record Hooke's law in the matrix form

$$\sigma = S\varepsilon \tag{24}$$

where σ and ε are total stress and strain matrices, and S is symmetric, then equation (23) becomes

$$\mathscr{U}^* = \iiint \delta\bar{\varepsilon}^T(\bar{\sigma} + \bar{\sigma}')\, dx\, dy\, dz \tag{25}$$

in conformity with equation (21) of [1]. However, in [1] only the linear approximations of the strain–displacement equations are substituted for $\bar{\varepsilon}$ into the variational strain energy, and this we can now recognize from [3] to be insufficient if the influences of steady state stress on structural stiffness are to be fully accommodated. Accordingly, we now consider the appropriate additional terms to be added to [1].

In terms of the matrix notation of [1], the strain displacement equations analogous to equations (12)–(17) of [3] but descriptive of the relationship between incremental strain matrix $\bar{\varepsilon}$ and the matrix $\bar{w}$ of incremental displacements $\bar{w}_1$, $\bar{w}_2$ and $\bar{w}_3$ can be written as

$$\bar{\varepsilon} = D\bar{w} + \tfrac{1}{2}(\bar{w}\Delta)^T\bar{w} \tag{26}$$

where the operators† D and Δ are defined in terms of local orthogonal coordinates ξ, η, and ζ by

$$D \triangleq \begin{bmatrix} \partial/\partial\xi & 0 & 0 \\ 0 & \partial/\partial\eta & 0 \\ 0 & 0 & \partial/\partial\zeta \\ \partial/\partial\eta & \partial/\partial\xi & 0 \\ 0 & \partial/\partial\zeta & \partial/\partial\eta \\ \partial/\partial\zeta & 0 & \partial/\partial\xi \end{bmatrix}$$

and

$$\Delta \triangleq \left[\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi} \quad \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta} \quad \frac{\partial}{\partial\zeta}\frac{\partial}{\partial\zeta} \quad 2\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} \quad 2\frac{\partial}{\partial\eta}\frac{\partial}{\partial\zeta} \quad 2\frac{\partial}{\partial\zeta}\frac{\partial}{\partial\xi}\right].$$

† These operators will be treated as matrices, but caution must be exercised in preserving a meaningful sequence of operations; in equation (26), for example, the operation $\bar{w}\,\Delta$ precedes the transposition, and such "products" as $\bar{w}_1\, 2(\partial/\partial\xi)(\partial/\partial\eta)$ are understood to mean the operator $2(\partial\bar{w}_1/\partial\xi)(\partial/\partial\eta)$.

Equation (26) is a nonlinear generalization of equation (14) of [1]. When this result is substituted into equation (25), and second degree terms in $\bar{w}$ are preserved when multiplied by the steady-state stress matrix $\bar{\sigma}'$, the result is the addition of the new term

$$\bar{y}^{*T}\bar{k}_{\Delta}\,\bar{y} \triangleq \bar{y}^{*T}\int W^T W \Delta\bar{\sigma}'\,dv\bar{y} \tag{27}$$

to the variational strain energy in equation (21) of [1], and correspondingly the new term $\bar{k}_{\Delta}\,\bar{y}$ to the expression for interaction force and torque in equation (34) of that paper. Here $\bar{y}$ is the $6\mathcal{N}$ by 1 matrix of incremental displacements of the $\mathcal{N}$ nodes of the finite element, and $\bar{k}_{\Delta}$ is the element *geometric stiffness matrix*. (The existence of this matrix is noted in [1], but no specific instructions for its construction are provided there.)

SUMMARY AND CONCLUSIONS

This addendum has had the objectives of expanding the scope of Ref. 1 and correcting a deficiency in that work which resulted from the neglect of certain potentially significant nonlinear terms in the strain–displacement equations. Even with the deeper appreciation of the subtleties of the mechanics of rotating finite elements reflected in this addendum, there remain many unanswered questions relating to the suitability of specific element models. The next step should be the detailed evaluation of the behavior of various finite element models of simple rotating structures, with the objective of evaluating the consequences of modeling decisions which are routine for nonrotating systems but potentially critical for rotating structures.

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